Active matter – Week 1: Stochastic motion at thermodynamic equilibrium
DRSTP Advanced Topics in Theoretical Physics (Spring 2025)

Yann-Edwin Keta
keta@lorentz.leidenuniv.nl

I. WHAT IS ACTIVE MATTER?

I.A. What is equilibrium and how to get out of it?

> Macroscopic systems often exhibit some "memory" of their recent history. A stirred cup of tea continues to swirl within the cup. Cold-worked steel maintains an enhanced hardness imparted by its mechanical treatment. But memory eventually fades. Turbulences damp out, internal strains yield to plastic flow, concentration inhomogeneities diffuse to uniformity. Systems tend to subside to very simple states, independent of their specific history.
> In some cases the evolution toward simplicity is rapid; in other cases it can proceed with glacial slowness. But *in all systems there is a tendency to evolve toward states in which the properties are determined by intrinsic factors and not by previously applied external influences. Such simple terminal states are, by definition, time independent. They are called equilibrium states.*
>
> – [1] (§ 1-5 – Thermodynamic equilibrium)

A working definition for this class is that equilibrium states are steady states which are time-reversal symmetric (*i.e.* their dynamics looks identical forward and backward) and in which there is no macroscopic flow (*e.g.* of matter, energy). We can then distinguish three general classes of nonequilibrium systems [2].

1. Systems relaxing towards equilibrium (*e.g.* thermal system adapting to its thermostat, glass [3]).

2. Systems with boundary conditions imposing steady currents (*e.g.* sheared liquid, metal rod between two thermostats).

3. Active matter.

I.B. Active matter

Active matter is a class of materials composed of *active particles*. These particles are self-driven units, individually capable of using available energy to generate forces [4, 5]. These forces constantly dissipate energy thus driving the system out of equilibrium.

This broad definition applies to a wide array of synthetic and living elements at all scales, from subcellular elements [6], to self-driven colloids [7], to birds [8] and humans [9]. Due to their continual generation of forces, these elements escape the rules of equilibrium statistical mechanics, and display a wealth of surprising phenomena which challenge our conceptions of equilibrium phases and dynamics.

The forces generated by active particles are stochastic[1]. In this class, we will be interested in understanding how these microscopic fluctuating forces lead to macroscopic dynamically correlated behaviours. We first need a framework to deal with such processes, and this framework is given by stochastic differential equations.

II. AN EXAMPLE OF STOCHASTIC DIFFERENTIAL EQUATION: LANGEVIN EQUATION

This section is largely based on [11] (§ 10.1 – The Langevin model).

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[1] "Stochastic" comes from the Ancient Greek stókhos "aim, guess" and is the property of being well-described by a random probability distribution [10].

II.A. Brownian motion

Brownian motion describes the erratic motion of a particle immersed in a fluid bath under the effect of the collisions it undergoes with the molecules of this fluid. This behaviour is exhibited by colloidal particles (*i.e.* glass or plastic beads, typically 1 µm in diameter) which are immersed in a fluid with matching density (in order not to sediment). As a matter of simplicity we will consider the 1-dimensional case.

We may write the Newtonian equation of motion for the position x of this particle

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{1a}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\zeta v + F(t), \tag{1b}$$

where m is its mass, ζ a viscous drag coefficient, and $F(t)$ is a random function – a *noise* – which represents the force arising from the collisions with the molecules of the fluid. Given a spherical colloid of radius R then $\zeta = 6\pi\mu R$, with μ the dynamical viscosity of the fluid, according to Stokes' law. This is the form of the equation introduced by Langevin [12] – and we thus commonly refer to (1) as the Langevin equation.

This first-order differential equation can be solved as[2]

$$v(t) = v(0)e^{-(\zeta/m)t} + \frac{1}{m}\int_0^t \mathrm{d}s\, e^{-(\zeta/m)(t-s)}F(s), \tag{2}$$

and assuming $v(t\to-\infty) = 0$,

$$v(t) = \frac{1}{m}\int_{-\infty}^t \mathrm{d}s\, e^{-(\zeta/m)(t-s)}F(s). \tag{3}$$

II.B. Stochastic force emerging from random collisions

Stationary process with zero average The bath is assumed to be at thermodynamic equilibrium, therefore (i) no instant or direction of time is privileged and (ii) it does not lead to a macroscopic flow. Condition (i) implies that $F(t)$ is modelled as a stationary stochastic process, *i.e.* $\langle F(t)\rangle$ does not depend on time and $\langle F(t)F(t')\rangle$ only depends on the time difference $t - t'$, and condition (ii) implies that

$$\langle F(t)\rangle = 0, \tag{4}$$

so that $\langle v\rangle = 0$.

Uncorrelated in time We introduce the autocorrelation function of the random force

$$g(\tau) = \langle F(t)F(t+\tau)\rangle\,, \tag{5a}$$

where condition (i) imposes that g is an even function of τ. There is in principle a characteristic time τ_c which describes the decay of g and corresponds to the typical time between two successive collisions with the fluid molecules. We will assume that τ_c is small compared to all other characteristic time scales, and write (5a) as

$$g(\tau) = 2\zeta^2 D\,\delta(\tau), \tag{5b}$$

where D is a diffusion constant and δ is the Dirac delta function[3]. This δ-peaked correlation function defines what is commonly referred to as a *white noise*[4].

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[2] This general integral solution (2) of the linear differential equation (1) can be found using Laplace transforms [13] (§ 12.5 – The Laplace transform – Physical applications, the Cauchy problem).

[3] Equations (1) and (5) define what is known as an Ornstein-Uhlenbeck process [14].

[4] Wiener–Khinchin theorem [11] (§ 1.10 – The Wiener–Khintchine theorem) states that the power spectral density of a stationary random process is given by the Fourier transform of its autocorrelation function. Force F has a δ-peaked autocorrelation function g, thus its Fourier transform is a constant. This flat spectral density defines the white noise by analogy with white light.

Normally distributed We assume that $F(t)$ is normally distributed. This may be justified using the central limit theorem: the force derives from the numerous collisions with the fluid molecules, therefore $F(t)$ may be thought as the superposition of a large number of identically distributed random functions.

Since (1) is a linear differential equation, $v(t)$ is also normally distributed.

Markovian character of v Since $F(t)$ is an uncorrelated white noise, all the realisations of F between times 0 and t are statistically identical, whatever the values of $F(0)$. Therefore, the knowledge of $v(0)$ is sufficient to characterise the statistics of $v(t)$ at times $t > 0$ using (2). This should be contrasted with the case where F has a finite correlation time: in this case, different values of $F(0)$ lead to statistically different realisations of $v(t)$ at times $t > 0$.

This defines a Markov process: the knowledge of only the present ($t = 0$) determines the future ($t > 0$). More information about Markov processes can be found in [15] (§ 3.2 – Markov process).

II.C. Fluctuation-dissipation theorem

It is important to note that both the random force $F(t)$ and the viscous drag $-\zeta v$ emerge from the interactions with the molecules of the fluid. This common origin is made apparent at equilibrium with fluctuation-dissipation relations which relate both these terms.

We compute the average kinetic energy from (3) and (5)

$$\begin{aligned}\left\langle v(t)^2\right\rangle &= \frac{1}{m^2}\int_{-\infty}^{t}\mathrm{d}s\int_{-\infty}^{t}\mathrm{d}s'\, e^{-(\zeta/m)(2t-s-s')}\,\left\langle F(s)F(s')\right\rangle \\ &= \frac{1}{m^2}\int_{-\infty}^{t}\mathrm{d}s\int_{-\infty}^{t}\mathrm{d}s'\, 2\zeta^2 D\, e^{-(\zeta/m)(2t-s-s')}\,\delta(s-s') \\ &= \frac{2\zeta^2 D}{m^2}\int_{-\infty}^{t}\mathrm{d}s\, e^{-2(\zeta/m)(t-s)} = \frac{\zeta D}{m}.\end{aligned} \tag{6}$$

Moreover, according to the equipartition theorem[5],

$$\frac{1}{2}m\left\langle v(t)^2\right\rangle = \frac{1}{2}k_B T, \tag{8}$$

where k_B is the Boltzmann constant and T is the temperature of the bath. Therefore (6) and (8) lead to the relation

$$D = \frac{k_B T}{\zeta}, \tag{9}$$

known as the Einstein relation, which relates the diffusion constant (the fluctuation) and the drag coefficient (the dissipation). This is the simplest form of what is known as the fluctuation-dissipation theorem. This theorem relates more broadly the equilibrium fluctuations of the system to its response to external perturbations; a more careful but more abstract introduction can be found in [17] (§ 8.5 – Fluctuation-dissipation theorem).

III. CONTINUUM DESCRIPTION OF A STOCHASTIC SYSTEM: FOKKER-PLANCK EQUATION

III.A. Kramers-Moyal expansion

This section is largely based on [11] (§ 11 – Brownian motion: the Fokker-Planck equation). We aim to derive from the Langevin equation (1) a differential equation for the probability $p(v, t)$ of observing velocity $v(t) = v$ at time t.

[5] Consider a system described by some degrees of freedom $\{x_1, \ldots, x_N\}$ and a Hamiltonian $H(x_1, \ldots, x_N)$. At equilibrium, the equipartition theorem gives the values of the following averages,

$$\left\langle x_i \frac{\partial}{\partial x_j} H\right\rangle = \delta_{ij} k_B T, \tag{7}$$

where k_B is the Boltzmann constant and T is the temperature of the system [16] (§ 6.4 – Equipartition theorem).

To this effect we write $p(v,t+\Delta t)$ as a function of $p(v,t)$ using conditional probabilities

$$p(v,t+\Delta t) = \int \mathrm{d}v'\, p(v,t+\Delta t|v',t)\, p(v',t) = \int \mathrm{d}w\, p(v,t+\Delta t|v-w,t)\, p(v-w,t), \tag{10}$$

with the idea of then taking the limit of $\Delta t \to 0$.

We derive $p(v+w,t+\Delta t|v,t)$ – a little simpler conceptually than $p(v,t+\Delta t|v-w,t)$ –, which is the probability that $v(t+\Delta) = v+w$ given that $v(t) = v$, from the Langevin equation. In order to do so, we may first integrate (1) between times t and $t+\Delta t$,

$$v(t+\Delta t) = v(t)e^{-(\zeta/m)\Delta t} + \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, e^{-(\zeta/m)(\Delta t - s)} F(t+s), \tag{11}$$

and write the difference $w = v(t+\Delta t) - v(t)$,

$$w = v(t)(e^{-(\zeta/m)\Delta t} - 1) + \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, e^{-(\zeta/m)(\Delta t - s)} F(t+s), \tag{12a}$$

$$\begin{aligned} w^2 = {} & v(t)^2(e^{-(\zeta/m)\Delta t} - 1)^2 + \frac{1}{m}\int_0^{\Delta t} \mathrm{d}s\, (e^{-(\zeta/m)\Delta t} - 1)e^{-(\zeta/m)(\Delta t - s)}\, v(t)F(t+s) \\ & + \frac{1}{m^2}\int_0^{\Delta t} \mathrm{d}s \int_0^{\Delta t} \mathrm{d}s' e^{-(\zeta/m)(2\Delta t - s - s')} F(t+s)F(t+s'). \end{aligned} \tag{12b}$$

Given the linearity of (1) and that $F(t)$ is a Gaussian process, it follows that $v(t)$ is a Gaussian process, and using (12a) that w is normally distributed. This means that $p(v+w,t+\Delta t|v,t)$ is uniquely determined by its first two moments given by the averages $\langle \ldots \rangle$ of (12) at fixed $v(t) = v$ over the different realisations of $F(t)$. While performing these averages it is noteworthy that

$$\langle F(t+s) \rangle = 0, \tag{13}$$

which is equivalent to (4), that for $s > 0$

$$\langle v(t)F(t+s) \rangle = 0, \tag{14}$$

which derives from $F(t)$ having zero correlation time, and that

$$\langle F(t+s)F(t+s') \rangle = 2\zeta^2 D\, \delta(s-s'), \tag{15a}$$

$$\begin{aligned} \int_0^{\Delta t} \mathrm{d}s \int_0^{\Delta t} \mathrm{d}s'\, e^{-(\zeta/m)(2\Delta t - s - s')}\, \langle F(t+s)F(t+s') \rangle &= 2\zeta^2 D \int_0^{\Delta t} \mathrm{d}s\, e^{-2(\zeta/m)(\Delta t - s)} \\ &= m\zeta D(1 - e^{-2(\zeta/m)\Delta t}), \end{aligned} \tag{15b}$$

where (15a) is equivalent to (5a). Collecting all these we get

$$M_1 = \langle w \rangle = v e^{-(\zeta/m)\Delta t} \underset{\Delta t \to 0}{=} -(\zeta/m)\Delta t\, v + \mathcal{O}(\Delta t^2), \tag{16a}$$

$$M_2 = \langle w^2 \rangle = v^2(e^{-(\zeta/m)\Delta} - 1)^2 + (\zeta/m)D(1 - e^{-2(\zeta/m)\Delta t}) \underset{\Delta t \to 0}{=} 2(\zeta/m)^2 D\Delta t + \mathcal{O}(\Delta t^2), \tag{16b}$$

which do not depend on the value of t but only on the difference of times Δt, consistently with stationarity. We thus write the corresponding multivariate normal distribution in d dimensions, at first order in Δt

$$\begin{aligned} p(v+w,t+\Delta t|v,t) &= \frac{1}{(2\pi(M_2 - M_1^2))^{d/2}} \exp\left(-\frac{1}{2}\frac{(w-M_1)^2}{M_2 - M_1^2}\right) \\ &\underset{\Delta t \to 0}{=} \frac{1}{(4\pi(\zeta/m)^2 D\Delta t)^{d/2}} \exp\left(-\frac{1}{2}\frac{(w+(\zeta/m)\Delta t v)^2}{4(\zeta/m)^2 D\Delta t}\right) \equiv \tilde{p}(w,v,\Delta t), \end{aligned} \tag{17}$$

where $\tilde{p}(w,v,\Delta t)$ now designates the probability of an increment w in velocity over time Δt from value v.

We may now rewrite (10) using the notation (17) and Taylor-expand the integrand as follows

$$\begin{aligned} p(v,t+\Delta t) &= \int \mathrm{d}w\, \tilde{p}(w,v-w,\Delta t)\, p(v-w,t) \\ &= \int \mathrm{d}w \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} w^n \frac{\partial^n}{\partial v^n}\Big[p(w,v,\Delta t)p(v,t)\Big] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial v^n}\left[\left(\int \mathrm{d}w\, w^n \tilde{p}(w,v,\Delta t)\right) p(v,t)\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial v^n}\Big[\, \langle w^n \rangle\, p(v,t)\Big], \end{aligned} \tag{18}$$

where $\langle \ldots \rangle$ designates an average with respect to distribution $\tilde{p}(w,v,\Delta t)$ (17). This expansion of $p(v,t+\Delta t)$ is known as the Kramers-Moyal expansion. The Fokker-Planck approximation consists in neglecting the terms of order $n \geq 3$, thus with the moments (16),

$$p(v,t+\Delta t) = p(v,t) + \Delta t \frac{\partial}{\partial v}\left(\frac{\zeta}{m} v p(v,t)\right) + \Delta t \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(2\frac{\zeta^2}{m^2} D p(v,t)\right), \tag{19}$$

which *finally* leads to the Fokker-Planck equation corresponding to our Langevin equation (1) by taking the limit $\Delta t \to 0$,

$$\frac{\partial}{\partial t} p(v,t) = \frac{\zeta}{m}\frac{\partial}{\partial v} v p(v,t) + \frac{\zeta^2}{m^2} D \frac{\partial^2}{\partial v^2} p(v,t). \tag{20}$$

III.B. General Fokker-Planck equation

A more general demonstration of the Fokker-Planck can be found in [18] (§ 2.2 – Fokker-Planck equations). Consider a quantity $\boldsymbol{a}$ described by a Langevin-like equation

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \boldsymbol{v}(\boldsymbol{a}) + \boldsymbol{F}(t), \tag{21}$$

where $\boldsymbol{F}(t)$ is Gaussian white noise with zero mean and variance

$$\langle \boldsymbol{F}(t) \otimes \boldsymbol{F}(t') \rangle = 2\, \mathbb{B}\, \delta(t-t'), \tag{22}$$

with $\mathbb{B}$ a constant covariance matrix. Under these conditions, the probability distribution $p(\boldsymbol{a},t)$ satisfies the following Fokker-Planck equation

$$\frac{\partial}{\partial t} p(\boldsymbol{a},t) = -\frac{\partial}{\partial \boldsymbol{a}} \cdot (\boldsymbol{v}(\boldsymbol{a}) p(\boldsymbol{a},t)) + \frac{\partial}{\partial \boldsymbol{a}} \cdot \left(\mathbb{B}\frac{\partial}{\partial \boldsymbol{a}} p(\boldsymbol{a},t)\right). \tag{23}$$

III.C. Smoluchowski equation and Boltzmann distribution

Langevin equations are often considered in the *overdamped* limit, where the inertial term

$$\frac{\mathrm{d}v}{\mathrm{d}t} \to 0, \tag{24}$$

which amounts to setting $m = 0$ in (1). This hypothesis is common in describing soft matter systems where dissipation dominates. It is similar to taking the Stokes limit of the Navier-Stokes equation in fluid dynamics as the Reynolds number $\mathrm{Re} \to 0$.

We can consider adding an energy potential $U(x)$, such that the stochastic equation of motion now reads

$$\zeta \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}U}{\mathrm{d}x} + F(t). \tag{25}$$

A special case is a harmonic potential $U(x) = (1/2)kx^2$ which leads to the exact same form as (1) where v is replaced by x. It is important to note once again that, with correlation function (5) and a time-independent $U(x)$, the knowledge of $x(0)$ is sufficient to characterise the statistics of $x(t)$ at times $t > 0$ since all the realisations of F between times 0 and t are statistically identical.

Using (23) and the Einstein relation (9) we find the following Fokker-Planck equation,

$$\frac{\partial}{\partial t} p(x,t) = \frac{1}{\zeta}\frac{\partial}{\partial x}\left(\frac{\mathrm{d}U}{\mathrm{d}x} p(x,t)\right) + D\frac{\partial^2}{\partial x^2} p(x,t) \tag{26a}$$

$$= -\frac{\partial}{\partial x}\left(-\frac{1}{\zeta} p(x,t)\frac{\partial}{\partial x}\left[k_B T \log p(x,t) + U\right]\right) = -\frac{\partial}{\partial x} J(x) \tag{26b}$$

which is known as Smoluchowski equation. It is noteworthy that in the case of a free system, *i.e.* with $U = 0$, (26a) is equivalent to the well-known diffusion equation,

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}. \tag{27}$$

Equation (26b) highlights that the temporal variations of probabilities p are balanced by the divergence of a flux[6]. As the system relaxes to equilibrium, this flux must cancel, as such we find the following solution

$$p(x, t \to \infty) = p_{\mathrm{ex}}(x) = \frac{1}{Z} e^{-\frac{U(x)}{k_B T}}, \tag{29a}$$

$$Z = \int \mathrm{d}x\, e^{-\frac{U(x)}{k_B T}}, \tag{29b}$$

where $p_{\mathrm{ex}}(x)$ is the Boltzmann distribution and Z the partition function which ensures it is normalised.

[6] We define the continuity equation of a quantity ρ as

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot \boldsymbol{j} + \sigma, \tag{28}$$

where is $\boldsymbol{j}$ its flux and σ is a source (or sink) term. This means that the quantity ρ at a given point in space varies over time either because there is an incoming flux of this quantity (first term) or it is created or annihilated (second term). In the case of $\sigma = 0$ it may be called a conservation equation that shows the balance between the variation in time and the incoming flux. This is the form that (26) takes in probability space, with the probability p being the conserved quantity and J being its flux.

IV. PROBLEMS

IV.A. Diffusion of a Brownian particle and Maxwell-Boltzmann distribution

Consider a particle with position x, mass m and drag coefficient ζ, which follow (1) and (5),

$$m\ddot{x} = -\zeta\dot{x} + F(t), \tag{30a}$$
$$\langle F(t)F(t')\rangle = 2\zeta k_B T\,\delta(t-t'). \tag{30b}$$

1. Compute the velocity autocorrelation function $\langle\dot{x}(t)\dot{x}(t')\rangle$ as a function of k_BT, m, t, and t'.

2. Compute the mean-squared displacement $\left\langle(x(t)-x(0))^2\right\rangle$ from the integral of the velocity autocorrelation function.

3. Show that $\left\langle(x(t)-x(0))^2\right\rangle \underset{t\to 0}{\sim} \left\langle\dot{x}(0)^2\right\rangle t^2$ and $\left\langle(x(t)-x(0))^2\right\rangle \underset{t\to\infty}{\sim} 2Dt$.

4. Show that the Maxwell-Boltzmann distribution,

$$p(\dot{x}) \propto \exp\left(-\frac{1}{2}\frac{m\dot{x}^2}{k_BT}\right), \tag{31}$$

is a steady-state solution of the Fokker-Planck equation (20) which corresponds to (30).

IV.B. Caldeira-Leggett model and generalised Langevin equation

This problem on the Caldeira-Leggett model is largely based on [11] (§ 10B – Brownian motion in a bath of oscillators) and [18] (§ 1.6 – Brownian motion in a harmonic oscillator heat bath). This model provides a microscopic description to conceptualise generalised Langevin equations (35). We will derive the fluctuation-dissipation relation (36) which applies for stochastic systems described by (35) [11] (§ 10A – The generalized Langevin model).

Consider a particle, described by its position x, at equilibrium with an ensemble of N oscillators, described by their positions x_i. We introduce the Hamiltonian of the system,

$$H = \frac{1}{2}m\dot{x}^2 + \sum_{i=1}^{N}\left[\frac{1}{2}m_i\dot{x}_i^2 + \frac{1}{2}m_i\omega_i^2\left(x_i - \frac{\gamma_i}{m_i\omega_i^2}x\right)^2\right], \tag{32}$$

where m and m_i are masses for the particle and the oscillators, ω_i are characteristic frequencies, and γ_i are coupling constants.

1. Using Hamilton's equations for the positions, x and x_i, and the momenta, $m\dot{x}$ and $m\dot{x}_i$, derive the equations of motion of the particle and the oscillators in the form of differential equations in $x(t)$ and $x_i(t)$.

2. Solve the equations of motion for $x_i(t)$ and give an expression

$$x_i(t) - \frac{\gamma_i}{m_i\omega_i^2}x(t) = \ldots \tag{33}$$

using only time t, initial values $x(0)$, $x_i(0)$ and $\dot{x}_i(0)$, and the *velocities* $\dot{x}(s)$ for times $0 \leq s \leq t$.

First hint: We recall that the differential equation

$$f''(t) + \omega^2 f(t) = g(t), \tag{34a}$$

admits the following solution[7]

$$f(t) = f(0)\cos(\omega t) + f'(0)\frac{\sin(\omega t)}{\omega} + \int_0^t \mathrm{d}s\, \frac{\sin(\omega(t-s))}{\omega} g(s). \tag{34b}$$

Second hint: An integration by part in (34b) enables to go from an integral on $g(s)$ to an integral on $\dot{g}(s)$.

3. Show that x satisfies a generalised Langevin equation,

$$m\ddot{x}(t) = -\int_0^t \mathrm{d}s\, \zeta(t-s)\,\dot{x}(s) + F(t), \tag{35}$$

and give the expressions of $\zeta(t-s)$ and $F(t)$ using time t and initial values $x(0)$, $x_i(0)$ and $\dot{x}_i(0)$.

4. Compute the average $\langle F(t)\rangle$ and correlations $\langle F(t)F(t')\rangle$ of the driving force as functions of t and t' only. Hint: averages $\langle ...\rangle$ over time and initial conditions $x(0)$, $x_i(0)$ and $\dot{x}_i(0)$ can be computed using the equipartition theorem.

5. Show that, at equilibrium, the generalised drag ζ and the driving noise F in the generalised Langevin equation (35) satisfy the following form of the fluctuation-dissipation theorem,

$$\langle F(t)F(t')\rangle = k_B T\, \zeta(t-t'). \tag{36}$$

IV.C. Active Onstein-Uhlenbeck particles and breakdown of equipartition

Active Onstein-Uhlenbeck particles are a common model of self-propelled particles. The first part of this problem is largely based on [19] (§ IV – Self-propelled particle in a harmonic potential) and the second part on [20] (§ Normal mode formulation).

Consider an overdamped particle with position x, driven by a self-propulsion force f and feeling a potential U. We write its equation of motion,

$$\zeta\dot{x} = -\frac{\partial}{\partial x}U + f, \tag{37a}$$

where ζ is a drag coefficient, and consider that p follows an Ornstein-Uhlenbeck process,

$$\tau_p \dot{f} = -f + \sqrt{2\zeta^2 v_0^2 \tau_p}\,\eta, \tag{37b}$$

where v_0 is the self-propulsion velocity, τ_p is the persistence time, and η is a Gaussian white noise with mean $\langle \eta(t)\rangle = 0$ and variance

$$\langle \eta(t)\eta(t')\rangle = \delta(t-t'). \tag{37c}$$

1. Derive the Fokker-Planck equation followed by $p(x, f, t)$.

2. Consider a harmonic potential, $U = \frac{1}{2}kx^2$ with k a spring constant. Show that the following Gaussian distribution is a steady-state solution of the Fokker-Planck equation,

$$p_{\mathrm{ss}}(x, f) \propto \exp(-ax^2 - bf^2 - cxf), \tag{38}$$

and give the values of a, b, and c.

3. Derive the marginal steady-state position distribution $p_{ss}(x)$. What is the mean energy of the elastic mode $\left\langle \frac{1}{2}kx^2 \right\rangle$? At what condition does this distribution describe a system at thermal equilibrium with temperature T?

[7] As mentioned in footnote 2, this can be shown using Laplace transforms.

4. Consider an ensemble of N active Ornstein-Uhlenbeck particles described by positions $\boldsymbol{x} = \{x_1, \ldots, x_N\}$, self-propulsion forces $\boldsymbol{f} = \{f_1, \ldots, f_N\}$, and Gaussian white noises $\boldsymbol{\eta} = \{\eta_1, \ldots, \eta_N\}$ which are independent random variables. We assume that these particles are coupled with linear forces,

$$-\frac{\partial}{\partial x_i} U = -\sum_{j=1}^{N} \mathbb{H}_{ij} x_j, \tag{39a}$$

$$\mathbb{H}_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j} = \text{cst}, \tag{39b}$$

where $\mathbb{H}$ is the Hessian matrix.
(a) Argue why $\mathbb{H}$ is diagonalisable in a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$ with $\boldsymbol{e}_m \cdot \boldsymbol{e}_n = \delta_{mn}$.
(b) Given an eigenvector $\boldsymbol{e}_n$ of $\mathbb{H}$, associated with eigenvalue κ_n, show that $\boldsymbol{e}_n \cdot \boldsymbol{x}$, $\boldsymbol{e}_n \cdot \boldsymbol{f}$, and $\boldsymbol{e}_n \cdot \boldsymbol{\eta}$ satisfy (37).
(c) What is the mean energy of each elastic eigenmode $\left\langle \frac{1}{2}\kappa_n(\boldsymbol{e}_n \cdot \boldsymbol{x})^2 \right\rangle$?
(d) At what conditions do these elastic eigenmodes follow equipartition?

REFERENCES

[1] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (Wiley, New York, 1985).
[2] M. E. Cates and J. Tailleur, Motility-Induced Phase Separation, Annual Review of Condensed Matter Physics **6**, 219 (2015).
[3] A. Cavagna, Supercooled liquids for pedestrians, Physics Reports **476**, 51 (2009), an excellent introduction to the physics of glasses.
[4] M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, M. Rao, and R. A. Simha, Hydrodynamics of soft active matter, Reviews of Modern Physics **85**, 1143 (2013).
[5] M. J. Bowick, N. Fakhri, M. C. Marchetti, and S. Ramaswamy, Symmetry, Thermodynamics, and Topology in Active Matter, Physical Review X **12**, 010501 (2022).
[6] T. Sanchez, D. T. N. Chen, S. J. DeCamp, M. Heymann, and Z. Dogic, Spontaneous motion in hierarchically assembled active matter, Nature **491**, 431 (2012).
[7] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot, and D. Bartolo, Emergence of macroscopic directed motion in populations of motile colloids, Nature **503**, 95 (2013).
[8] A. Cavagna and I. Giardina, Bird Flocks as Condensed Matter, Annual Review of Condensed Matter Physics **5**, 183 (2014).
[9] J. L. Silverberg, M. Bierbaum, J. P. Sethna, and I. Cohen, Collective Motion of Humans in Mosh and Circle Pits at Heavy Metal Concerts, Physical Review Letters **110**, 228701 (2013).
[10] Wikipedia contributors, Stochastic — Wikipedia, the free encyclopedia ().
[11] N. Pottier, *Nonequilibrium Statistical Physics: Linear Irreversible Processes*, Oxford Graduate Texts (Oxford University Press, Oxford, 2010).
[12] P. Langevin, Sur la théorie du mouvement brownien, Comptes-rendus de l'Académie des sciences **146**, 301 (1908).
[13] W. Appel and E. Kowalski, *Mathematics for Physics and Physicists* (Princeton University Press, Princeton, N.J, 2007).
[14] Wikipedia contributors, Ornstein–uhlenbeck process — Wikipedia, the free encyclopedia ().
[15] C. W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry, and the Natural Sciences*, 3rd ed., Springer Series in Synergetics (Springer, 2004).
[16] K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, 1987).
[17] D. Chandler, *Introduction to Modern Statistical Mechanics*, 1st ed. (Oxford University Press, New York, 1987).
[18] R. Zwanzig, *Nonequilibrium Statistical Mechanics* (Oxford University Press, Oxford ; New York, 2001).
[19] G. Szamel, Self-propelled particle in an external potential: Existence of an effective temperature, Physical Review E **90**, 012111 (2014).
[20] S. Henkes, K. Kostanjevec, J. M. Collinson, R. Sknepnek, and E. Bertin, Dense active matter model of motion patterns in confluent cell monolayers, Nature Communications **11**, 1405 (2020).

V. SOLUTIONS

V.A. Diffusion of a Brownian particle and Maxwell-Boltzmann distribution

1. We consider $t' > t$ without loss of generality, then

$$\dot{x}(t') = e^{-(\zeta/m)(t'-t)}\dot{x}(t) + \frac{1}{m}\int_t^{t'} \mathrm{d}s\, e^{-(\zeta/m)(t'-s)}F(s), \tag{40}$$

and we write the correlation function,

$$\langle \dot{x}(t)\dot{x}(t')\rangle = e^{-(\zeta/m)(t'-t)}\left\langle \dot{x}(t)^2\right\rangle + \frac{1}{m}\int_t^{t'} \mathrm{d}s\, e^{-(\zeta/m)(t'-s)}\left\langle \dot{x}(t)F(s)\right\rangle, \tag{41}$$

where the second term cancels because F is a white noise thus the present velocity is uncorrelated with the future values of the force (14), and where the equipartition theorem dictates $\left\langle \dot{x}(t)^2\right\rangle = k_BT/m$. It should be noted that in the hypothesis $t' < t$, the sign in the exponential would be inverted, therefore we write

$$\langle \dot{x}(t)\dot{x}(t')\rangle = \frac{k_BT}{m}e^{-(\zeta/m)|t'-t|}, \tag{42}$$

for any t and t'.

2. We write the mean-squared displacement in integral form,

$$\left\langle (x(t)-x(0))^2\right\rangle = \left\langle \left(\int_0^t \mathrm{d}s\,\dot{x}(s)\right)^2\right\rangle = \int_0^t \mathrm{d}s\int_0^t \mathrm{d}s'\,\langle \dot{x}(s)\dot{x}(s')\rangle\,. \tag{43}$$

We use (42), and separate the integrals with $s < s'$ and $s > s'$,

$$\begin{aligned}
\left\langle (x(t)-x(0))^2\right\rangle &= \frac{k_BT}{m}\int_0^t \mathrm{d}s\left[\int_0^s \mathrm{d}s' e^{-(\zeta/m)(s-s')} + \int_s^t \mathrm{d}s' e^{-(\zeta/m)(s'-s)}\right]\\
&= \frac{k_BT}{m}\frac{m}{\zeta}\int_0^t \mathrm{d}s\left[1-e^{-(\zeta/m)s}+1-e^{-(\zeta/m)(t-s)}\right]\\
&= \frac{k_BT}{m}\frac{m}{\zeta}\left[2t-\frac{m}{\zeta}\left(1-e^{-(\zeta/m)t}+1-e^{-(\zeta/m)t}\right)\right]\\
&= 2D\left[t+\frac{m}{\zeta}\left(e^{-(\zeta/m)t}-1\right)\right]
\end{aligned} \tag{44}$$

where we have used the Einstein relation (9) in the last line.

3. We Taylor-expand the mean-squared displacement (44) at small times,

$$\left\langle (x(t)-x(0))^2\right\rangle \underset{t\to 0}{=} 2D\left[t+\frac{m}{\zeta}\left(1-\frac{\zeta}{m}t+\frac{1}{2}\frac{\zeta^2}{m^2}t^2+\mathcal{O}(t^3)-1\right)\right] \underset{t\to 0}{\sim} \frac{D\zeta}{m}t^2 = \left\langle v(0)^2\right\rangle t^2, \tag{45}$$

where the last equality uses the autocorrelation function (42) and the Einstein relation (9), and we take the limit at large times,

$$\left\langle (x(t)-x(0))^2\right\rangle \underset{t\to\infty}{\sim} 2Dt. \tag{46}$$

4. We have derived the Fokker-Planck equation (20) corresponding to our model (30),

$$\frac{\partial}{\partial t}p(\dot{x},t) = -\frac{\zeta}{m}\frac{\partial}{\partial \dot{x}}\dot{x}p(\dot{x},t) + \frac{\zeta^2}{m^2}D\frac{\partial^2}{\partial \dot{x}^2}p(\dot{x},t), \tag{47}$$

such that the steady state $t \to \infty$ solution verifies

$$0 = \frac{\partial}{\partial \dot{x}} \left(\dot{x} p(\dot{x}) + \frac{k_B T}{m} \frac{\partial}{\partial \dot{x}} p(\dot{x}) \right) = \frac{\partial}{\partial \dot{x}} \left(p(\dot{x}) \frac{\partial}{\partial \dot{x}} \left[\frac{k_B T}{m} \log p(\dot{x}) + \frac{1}{2} \dot{x}^2 \right] \right), \tag{48}$$

where we have once again used Einstein relation (9). It is clear from this expression that the Maxwell-Boltzmann distribution $p(\dot{x}) \propto \exp(-\frac{1}{2}\frac{m\dot{x}^2}{k_B T})$ lays a constant term between square brackets, therefore its derivative cancels and it is a steady-state solution of the Fokker-Planck equation.

V.B. Caldeira-Leggett model and generalised Langevin equation

1. We write Hamilton's equations,

$$\dot{x} = \frac{1}{m} \frac{\partial H}{\partial \dot{x}}, \qquad m\ddot{x} = -\frac{\partial H}{\partial x} = \sum_{i=1}^{N} \gamma_i \left(x_i - \frac{\gamma_i}{m_i \omega_i^2} x \right), \tag{49a}$$

$$\dot{x}_i = \frac{1}{m_i} \frac{\partial H}{\partial \dot{x}_i}, \qquad m_i \ddot{x}_i = -\frac{\partial H}{\partial x_i} = m_i \omega_i^2 \left(\frac{\gamma_i}{m_i \omega_i^2} x - x_i \right). \tag{49b}$$

2. We rewrite (49b) in the form of (34a),

$$\ddot{x}_i + \omega_i^2 x_i = \frac{\gamma_i}{m_i} x, \tag{50}$$

and use the solution (34b),

$$x_i(t) = x_i(0) \cos(\omega_i t) + \dot{x}_i(0) \frac{\sin(\omega_i t)}{\omega_i} + \int_0^t \mathrm{d}s \, \frac{\sin(\omega_i (t-s))}{\omega_i} \frac{\gamma_i}{m_i} x(t). \tag{51}$$

This expression depends on the positions $x(s)$ for times $0 \leq s \leq t$. We perform an integration by part to obtain an expression which depends on the *velocities* $\dot{x}(s)$,

$$x_i(t) = x_i(0) \cos(\omega_i t) + \dot{x}_i(0) \frac{\sin(\omega_i t)}{\omega_i} + \left[\frac{\cos(\omega_i (t-s))}{\omega_i^2} \frac{\gamma_i}{m_i} x(s) \right]_{s=0}^{s=t} - \int_0^t \mathrm{d}s \, \frac{\cos(\omega_i (t-s))}{\omega_i^2} \frac{\gamma_i}{m_i} \dot{x}(s), \tag{52}$$

which we can rewrite to reveal (33),

$$x_i(t) - \frac{\gamma_i}{m_i \omega_i^2} x(t) = \left(x_i(0) - \frac{\gamma_i}{m_i \omega_i^2} x(0) \right) \cos(\omega_i t) + \dot{x}_i(0) \frac{\sin(\omega_i t)}{\omega_i} - \int_0^t \mathrm{d}s \, \frac{\gamma_i \cos(\omega_i (t-s))}{m_i \omega_i^2} \dot{x}(s). \tag{53}$$

3. We combine (49a) and (53),

$$m\ddot{x} = -\int_0^t \mathrm{d}s \left(\sum_{i=1}^{N} \frac{\gamma_i^2}{m_i \omega_i^2} \cos(\omega_i (t-s)) \right) \dot{x}(s) + \sum_{i=1}^{N} \gamma_i \left(\left(x_i(0) - \frac{\gamma_i}{m_i \omega_i^2} x(0) \right) \cos(\omega_i t) + \dot{x}_i(0) \frac{\sin(\omega_i t)}{\omega_i} \right), \tag{54}$$

and identify

$$\zeta(t-s) = \sum_{i=1}^{N} \frac{\gamma_i^2}{m_i \omega_i^2} \cos(\omega_i (t-s)), \tag{55a}$$

$$F(t) = \sum_{i=1}^{N} \gamma_i \left(\left(x_i(0) - \frac{\gamma_i}{m_i \omega_i^2} x(0) \right) \cos(\omega_i t) + \dot{x}_i(0) \frac{\sin(\omega_i t)}{\omega_i} \right). \tag{55b}$$

4. $F(t)$ is a sum of cosines and sines, therefore its time-average cancels,

$$\langle F(t) \rangle = 0. \tag{56}$$

We write the correlator,

$$\langle F(t)F(t')\rangle = \left\langle \sum_{i=1}^{N}\gamma_i\left(\left(x_i(0)-\frac{\gamma_i}{m_i\omega_i^2}x(0)\right)\cos(\omega_i t)+\dot{x}_i(0)\frac{\sin(\omega_i t)}{\omega_i}\right)\right.$$
$$\left.\times\sum_{i=1}^{N}\gamma_i\left(\left(x_j(0)-\frac{\gamma_j}{m_j\omega_j^2}x(0)\right)\cos(\omega_j t')+\dot{x}_i(0)\frac{\sin(\omega_j t')}{\omega_j}\right)\right\rangle, \tag{57}$$

where the cross terms $i \neq j$ involve products of cosines and sines with different frequencies and whose time-average thus cancels,

$$\langle F(t)F(t')\rangle = \sum_{i=1}^{N}\gamma_i^2\left(\left\langle\left(x_i(0)-\frac{\gamma_i}{m_i\omega_i^2}x(0)\right)^2\right\rangle\langle\cos(\omega_i t)\cos(\omega_i t')\rangle+\left\langle\dot{x}_i(0)^2\right\rangle\frac{\langle\sin(\omega_i t)\sin(\omega_i t')\rangle}{\omega_i^2}\right). \tag{58}$$

We compute the ensemble averages over the initial conditions using the equipartition theorem given the Hamiltonian (32),

$$\left\langle m_i\dot{x}_i^2\right\rangle = k_B T, \tag{59a}$$
$$\left\langle m_i\omega_i^2\left(x_i-\frac{\gamma_i}{m_i\omega_i^2}x\right)^2\right\rangle = k_B T, \tag{59b}$$

which lead to

$$\langle F(t)F(t')\rangle = \sum_{i=1}^{N}\frac{\gamma_i^2}{m_i\omega_i^2}k_B T\,\langle\cos(\omega_i t)\cos(\omega_i t')+\sin(\omega_i t)\sin(\omega_i t')\rangle = \sum_{i=1}^{N}\frac{\gamma_i^2}{m_i\omega_i^2}k_B T\cos(\omega_i(t-t')). \tag{60}$$

5. We identify (36) from (55a) and (60).

V.C. Active Onstein-Uhlenbeck particles and breakdown of equipartition

1. We use (23) and write the Fokker-Planck equation corresponding to the equations of motion (37),

$$\frac{\partial}{\partial t}p(x,f,t) = -\frac{1}{\zeta}\frac{\partial}{\partial x}\left(\left[-\frac{\partial}{\partial x}U+f\right]p(x,f,t)\right)+\frac{1}{\tau_p}\frac{\partial}{\partial f}\left(fp(x,f,t)\right)+\frac{\zeta^2 v_0^2}{\tau_p}\frac{\partial^2}{\partial f^2}p(x,f,t). \tag{61}$$

2. We take the steady-state limit $\partial_t p = 0$ and insert the Gaussian ansatz p_{s} (38) into the Fokker-Planck equation (61),

$$\begin{aligned}
0 &= --\frac{1}{\zeta}\frac{\partial}{\partial x}\left([-kx+f]p_{\mathrm{ss}}\right)+\frac{1}{\tau_p}\frac{\partial}{\partial f}\left(fp_{\mathrm{ss}}\right)+\frac{\zeta^2 v_0^2}{\tau_p}\frac{\partial^2}{\partial f^2}p_{\mathrm{ss}}\\
&= \frac{k}{\zeta}p_{\mathrm{ss}}+\frac{(kx-f)}{\zeta}\frac{\partial}{\partial x}p_{\mathrm{ss}}+\frac{1}{\tau_p}p_{\mathrm{ss}}+\frac{f}{\tau_p}\frac{\partial}{\partial f}p_{\mathrm{ss}}+\frac{\zeta^2 v_0^2}{\tau_p}\frac{\partial^2}{\partial f^2}p_{\mathrm{ss}}\\
&= p_{\mathrm{ss}}\left[\frac{k}{\zeta}+\frac{kx-f}{\zeta}(-2ax-cf)+\frac{1}{\tau_p}+\frac{f}{\tau_p}(-2bf-cx)+\frac{\zeta^2 v_0^2}{\tau_p}(-2b+(2bf+cx)^2)\right]\\
&= p_{\mathrm{ss}}\Big[\left(\frac{k}{\zeta}+\frac{1}{\tau_p}-\frac{2\zeta^2 v_0^2}{\tau_p}b\right)\\
&\qquad+\left(-\frac{2ak}{\zeta}+\frac{\zeta^2 v_0^2}{\tau_p}c^2\right)x^2\\
&\qquad+\left(\frac{1}{\zeta}c-\frac{2}{\tau_p}b+\frac{4\zeta^2 v_0^2}{\tau_p}b^2\right)f^2\\
&\qquad+\left(-\frac{k}{\zeta}c+\frac{2}{\zeta}a-\frac{1}{\tau_p}c+\frac{4\zeta^2 v_0^2}{\tau_p}bc\right)fx\Big].
\end{aligned} \tag{62}$$

Our Gaussian ansatz p_{ss} (38) thus satisfies our steady-state Fokker-Planck equation (61) if all the terms between parentheses in the last line of (62) cancel. We assume that all parameters are non-zero $\zeta, k, \tau_p, v_0^2 \neq 0$, then we infer

$$\frac{k}{\zeta} + \frac{1}{\tau_p} - \frac{2\zeta^2 v_0^2}{\tau_p} b = 0 \Rightarrow b = \frac{1}{2\zeta^2 v_0^2}\left(1 + \frac{k\tau_p}{\zeta}\right), \tag{63a}$$

$$\frac{c}{\zeta} - \frac{2}{\tau_p} b + \frac{4\zeta^2 v_0^2}{\tau_p} b^2 = 0 \Rightarrow c = \frac{2\zeta}{\tau_p} b\left(1 - 2\zeta^2 v_0^2 b\right) = -\frac{k}{\zeta^2 v_0^2}\left(1 + \frac{k\tau_p}{\zeta}\right), \tag{63b}$$

$$-\frac{2ak}{\zeta} + \frac{\zeta^2 v_0^2}{\tau_p} c^2 = 0 \Rightarrow a = \frac{\zeta}{2k}\frac{\zeta^2 v_0^2}{\tau_p} c^2 = \frac{k}{2\zeta v_0^2 \tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right)^2, \tag{63c}$$

where we have used only three out of the four terms; we checked that the last also cancels,

$$\begin{aligned}
-\frac{k}{\zeta}c + \frac{2}{\zeta}a - \frac{1}{\tau_p}c + \frac{4\zeta^2 v_0^2}{\tau_p} bc &= -c\left(\frac{1}{\tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right) - \frac{4\zeta^2 v_0^2}{\tau_p} b\right) + \frac{2}{\zeta}a \\
&= \frac{k}{\zeta^2 v_0^2}\left(1 + \frac{k\tau_p}{\zeta}\right)\left(\frac{1}{\tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right) - \frac{4\zeta^2 v_0^2}{\tau_p}\frac{1}{2\zeta^2 v_0^2}\left(1 + \frac{k\tau_p}{\zeta}\right)\right) + \frac{2}{\zeta}\frac{k}{2\zeta v_0^2 \tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right)^2 \\
&= 0.
\end{aligned} \tag{63d}$$

3. We define normalisation factors,

$$\mathcal{N}_{xf} = \int \mathrm{d}x \int \mathrm{d}f\ \exp(-ax^2 - bf^2 - cxf), \tag{64a}$$

$$\mathcal{N}_f = \int \mathrm{d}f\ \exp(-bf^2), \tag{64b}$$

then the marginal distribution is

$$\begin{aligned}
p_{ss}(x) &= \frac{1}{\mathcal{N}_{xf}} \int \mathrm{d}f\ \exp(-ax^2 - bf^2 - cxf) \\
&= \frac{1}{\mathcal{N}_{xf}} \exp(-ax^2) \int \mathrm{d}f\ \exp\left(-b\left(f + \frac{1}{2}\frac{c}{b}x\right)^2\right) \exp\left(\frac{1}{4}\frac{c^2}{b}x^2\right) \\
&= \frac{\mathcal{N}_f}{\mathcal{N}_{fx}} \exp\left(-\frac{1}{2}\left(2a - \frac{1}{2}\frac{c^2}{b}\right)x^2\right).
\end{aligned} \tag{65}$$

Given the normal distribution (65), we directly read $\langle x \rangle = 0$ and the inverse variance

$$\begin{aligned}
\left\langle x^2 \right\rangle^{-1} &= 2a - \frac{1}{2}\frac{c^2}{b} \\
&= \frac{k}{\zeta v_0^2 \tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right)^2 - \frac{1}{2}\frac{\frac{k^2}{\zeta^4 v_0^4}\left(1 + \frac{k\tau_p}{\zeta}\right)^2}{\frac{1}{2\zeta^2 v_0^2}\left(1 + \frac{k\tau_p}{\zeta}\right)} \\
&= \frac{k}{\zeta v_0^2 \tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right)\left(\frac{1}{\tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right) - \frac{k}{\zeta}\right) \\
&= \frac{k}{\zeta v_0^2 \tau_p}\left(1 + \frac{k\tau_p}{\zeta}\right).
\end{aligned} \tag{66}$$

where we have used the values of the parameters a, b, c (63). We conclude that the mean energy of the elastic mode is

$$\left\langle \frac{1}{2}kx^2 \right\rangle = \frac{1}{2}\frac{\zeta v_0^2 \tau_p}{1 + k\tau_p/\zeta}. \tag{67}$$

At thermal equilibrium at temperature T the equipartition theorem would hold, which leads to the following relation

$$\frac{\zeta v_0^2 \tau_p}{1 + k\tau_p/\zeta} = k_B T. \tag{68}$$

4. (a) We have $\mathbb{H}_{ij} = \partial^2_{x_i x_j} U = \partial^2_{x_j x_i} U = \mathbb{H}_{ji}$, therefore $\mathbb{H}$ is an $N \times N$ symmetric real matrix, it follows from the spectral theorem that is diagonalisable in an orthogonal basis. We pick one such basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$ where all vectors are normalised and define $\{\kappa_1, \ldots, \kappa_N\}$ the corresponding eigenvalues.
(b) We derive the equation of motion for $\boldsymbol{e}_n \cdot \boldsymbol{x}$,

$$\begin{aligned}
\zeta \frac{\partial}{\partial t} \boldsymbol{e}_n \cdot \boldsymbol{x} &= \sum_{i=1}^{N} \zeta e_{n,i} \dot{x}_i \\
&= \sum_{i=1}^{n} e_{n,i} \left(- \sum_{j=1}^{n} \mathbb{H}_{ij} x_j + f_i \right) \\
&= - \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \mathbb{H}_{ij} e_{n,i} \right) x_j + \sum_{i=1}^{N} e_{n,i} f_i \\
&= - \sum_{i=1}^{N} \kappa_n e_{n,i} x_i + \sum_{i=1}^{N} e_{n,i} f_i \\
&= -\kappa_n \, \boldsymbol{e}_n \cdot \boldsymbol{x} + \boldsymbol{e}_n \cdot \boldsymbol{f},
\end{aligned} \tag{69a}$$

consistently with (37a), the equation of motion for $\boldsymbol{e}_n \cdot \boldsymbol{f}$,

$$\tau_p \frac{\partial}{\partial t} \boldsymbol{e}_n \cdot \boldsymbol{f} = \sum_{i=1}^{N} \tau_p e_{n,i} \dot{f}_i = - \sum_{i=1} e_{n,i} f_i + \sum_{i=1}^{N} e_{n,i} \sqrt{2\zeta^2 v_0^2 \tau_p}\, \eta_i = -\boldsymbol{e}_n \cdot \boldsymbol{f} + \sqrt{2\zeta^2 v_0^2 \tau_p}\, \boldsymbol{e}_n \cdot \boldsymbol{\eta}, \tag{69b}$$

consistently with (37b), and finally the variance of the noise $\boldsymbol{e}_n \cdot \boldsymbol{\eta}$,

$$\langle (\boldsymbol{e}_n \cdot \boldsymbol{\eta}(t))(\boldsymbol{e}_n \cdot \boldsymbol{\eta}(t')) \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} e_{n,i} e_{n,j} \langle \eta_i(t) \eta_j(t') \rangle = \sum_{i=1}^{N} e_{n,i}^2 \delta(t - t') = \delta(t - t') \tag{69c}$$

consistently with (37c), where we have cancelled cross terms $i \neq j$ due to the independence of white noises η_i and where the last equality derives from the normalisation of the eigenvectors.
(c) We use an analogy with the previous derivation. Indeed we just showed that $\boldsymbol{e}_n \cdot \boldsymbol{x}$ follows the same dynamics as the previously studied x with the potential $U = \frac{1}{2}\kappa_n x^2$. Therefore we can directly use (67) to write

$$\left\langle \frac{1}{2} \kappa_n (\boldsymbol{e}_n \cdot \boldsymbol{x})^2 \right\rangle = \frac{1}{2} \frac{\zeta v_0^2 \tau_p}{1 + \kappa_n \tau_p/\zeta}. \tag{70}$$

(d) We see that the mean energy of the elastic modes depend on the values of κ_n if $\tau_p \neq 0$. Therefore for (70) to follow equipartition we should either have that all κ_n are equal or that $\tau_p = 0$. In the first case, this means that $\mathbb{H}$ is the identity matrix and thus that particles do not interact with each other but are all in their own harmonic potential. In the second case this means that each particle would perform pure diffusion when they do not interact. We have actually just shown how the dynamics of an isolated active particle can be described at equilibrium by defining an effective temperature (68) but that this description fails when the particles are in interaction. We will come back to this in the next classes.